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ON THE EXTENSION OF CROTTI'S THEOREM TO THE THEORY OF FINITE ELASTIC DISPLACEMENTS

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Abstract—This paper deals with the extension of the classical Crotti's theorem to the theory of finite elastic displacements. In its classical formulation Crotti's theorem is based on the complementary work of deformation of the so-called internal forces and is restricted to Eulerian treatment of the geometrically linearized theory of elasticity. In order to investigate the conditions for the extension of Crotti's theorem to the theory of finite elastic displacements, a Lagrangian description of deformation is assumed and the classical Green–Lagrange tensor is adopted as measure of strain. The internal forces are expressed in terms of the Kirchhoff–Trefftz stress tensor, which is energetically conjugate to the Green–Lagrange strain tensor. The expression corresponding to the complementary work of applied forces in terms of internal stresses differs from the integral of the complementary strain energy density by the addition of a term yielded by the product of displacement gradients which, of course, vanishes in the geometrical linearized theory. The conditions required so that this expression is a perfect differential and is therefore employable in the extension of Crotti's theorem to the finite displacement theory of elasticity are then determined. Furthermore, by a general technique of calculus of variations it is shown that the compatibility conditions of the problem are implicitly embedded in this extended formulation of the theorem, as happens in the classical one. Finally, an application to a very simple example in the theory of structures is discussed.

1. INTRODUCTION

Crotti's theorem (Crotti, 1878) is the second of the classical theorems of structural mechanics on the derivatives of deformation work, direct and complementary, of elastic systems.

Cotterill–Castigliano's theorem (Cotterill, 1865; Castigliano, 1879) is the first of these theorems and involves the direct work of deformation of internal forces, i.e. the work given by the product of forces as intensive factors by displacements as extensive factors. It is therefore based on the concept of internal elastic energy and it is expressed by the derivative of a function of the displacement field only, which is implicitly a condition of equilibrium of the problem. Cotterill–Castigliano's theorem is linked to the stationary value of the total potential energy and is therefore valid in the geometrically linearized as well as in the finite displacement theory of elasticity.

Crotti's theorem, on the contrary, involves the complementary work of deformation of internal forces, i.e. the work given by the product of displacements as intensive factors by forces as extensive factors.

Its classical expression is given by the formula

$$\frac{\partial U_c}{\partial P_p} = \Delta_p \tag{1}$$

where the complementary internal energy U_c can be expressed in terms of the Eulerian stress tensor S and Δ_p is the generalised displacement corresponding to the generalised force P_p.

In this form Crotti's theorem is linked to the theorem of virtual forces and results, conversely to Cotterill–Castigliano's theorem, in a condition of compatibility (Dorn and Schild, 1956). Its validity is however restricted to the geometrically linearized theory of elasticity.

The difficulties in the extension of Crotti's theorem to the finite displacement theory of elasticity arise from the fact that on account of the non-linearity of the problem it is not possible to formulate a complementary energy principle which yields the system of compatibility equations by means of variation of the Eulerian stress field.

Recently the extension of Crotti's theorem to the geometrically non-linear theory of elasticity has become once again subject for discussion among structural engineers on account of a proposal formulated by El Naschie *et al.* (El Naschie, 1988; El Naschie and Al Athel, 1989; El Naschie, 1990). This proposal is based on the application of the Euler–Legendre transformation to the generalised applied force–displacement space according to Engesser's definition of complementary energy (Engesser, 1889). However, it is clear that this procedure cannot fulfil the real extension of the theorem to the finite displacement theory of elasticity because it is not able to yield the compatibility conditions of the problem through the derivatives of a function involving the internal stress field.

The aim of the present paper is to re-state the problem in a general form and show how it is possible to get such a result starting from the direct investigation of the correspondence between the complementary work of applied forces and a term involving internal stresses and strains. In order to do so it is necessary to make reference to a Lagrangian description of transformation, and it appears opportune to make use of the Green–Lagrange strain tensor and the Kirchhoff–Trefftz stress tensor, which are energetically conjugate.

The expression obtained is obviously different from the one holding in the geometrically linearized theory on account of the non-linearity of the problem. The term corresponding to the complementary work of external forces is not simply the Euler–Legendre transformation of the direct work of the internal stress field. Nevertheless it is shown that this term can be rearranged so that a perfect differential can be recognisable under certain conditions. As one would have expected, these conditions are not entirely derivable from thermodynamical statements.

On one hand it is thus possible to keep 'memory' of the transformation from initial to final configurations but, on the other hand, the description of the stress field becomes dependent on the state of deformation.

To clarify the compatibility conditions of the problem implicitly embedded in the expression obtained, it is advantageous to make reference to the techniques developed for multiple-field variational principles (Reissner, 1953; Baldacci, 1967; Fraeijs de Veubeke, 1972; Washizu, 1975; Oden and Reddy, 1983). Therefore we employ the Piola–Signorini form of the equilibrium equations in the Lagrangian scheme to render explicit the condition of compatibility contained in the expression developed here.

Finally, the result obtained in generality for a class of elastic bodies is applied to a very simple example in the theory of structures. The example is the case of an elastically hinged beam and we operate in the generalised internal force–displacement space.

2. LAGRANGIAN DESCRIPTION OF TRANSFORMATION

Referring both the initial and the strained configuration of an elastic body to the same Cartesian frame, let us indicate the coordinates of a material point as $\mathbf{x}$ in the initial configuration and as $\mathbf{y}$ in the final configuration. The displacement field is given by

$$\mathbf{u}(\mathbf{x}) = \mathbf{y}(\mathbf{x}) - \mathbf{x}. \tag{2}$$

For the sake of clarity we will define gradients in $\mathbf{x}$ by ∇ and gradients in $\mathbf{y}$ by $\hat{\nabla}$ (cf. Appendix for a brief account of the scheme of notation adopted). Moreover, $\mathbf{y}$ and $\mathbf{u}$ are supposed to be class C^N ($N \geqslant 2$) vector fields on the region V_0 occupied by the body in the initial configuration.

We also suppose that the change in configuration gives origin to an invertible transformation, so that if the elements of the Jacobian matrix are

$$\mathbf{F} = \nabla \mathbf{y} \tag{3}$$

the elements of its inverse are, on the other hand,

$$\mathbf{F}^{-1} = \hat{\nabla}\mathbf{x}. \tag{4}$$

The infinitesimal volume element $\mathrm{d}V$ in the strained configuration is given by

$$\mathrm{d}V = J\,\mathrm{d}V_0 \tag{5}$$

where J is the determinant of the Jacobian matrix; in the nature of a physical assumption, it can be supposed that $J > 0$ everywhere.

The following relationship holds between the oriented boundary surface elements in the final and initial configuration:

$$\mathbf{n}\,\mathrm{d}A = J\mathbf{F}^{-T}\cdot\mathbf{n}_0\mathrm{d}A_0 \tag{6}$$

$\mathbf{n}$ and $\mathbf{n}_0$ being the outward unit normals on the boundary of the region occupied by the body, respectively, in the final and initial configuration.

In this framework we can assume as measure of strain in the region V_0 the Green–Lagrange strain tensor $\mathbf{D}$:

$$2\mathbf{D} = (\nabla\mathbf{u}+\nabla\mathbf{u}^T+\nabla\mathbf{u}^T\cdot\nabla\mathbf{u}) \quad \text{in } V_0. \tag{7}$$

In the following we will assume equations (7) to be on the whole a set of compatibility equations. In fact if we assign an arbitrary tensor field $\mathbf{D}$ on V_0 the non-linear set of differential equations (7) in the unknown vector field $\mathbf{u}$ (or, what is equivalent, in $\mathbf{y}$) does not necessarily admit a solution. However, it can be demonstrated (Murnaghan, 1951; Novozhilov, 1961) that if the components of the tensor field $\mathbf{D}$ satisfy some additional conditions, called conditions of compatibility, the system of differential equations (7) admits a vector field solution $\mathbf{u}$. Under certain circumstances the uniqueness is also assured.

The conditions of compatibility are equivalent to the vanishing of the Riemann–Christoffel tensor based on $\mathbf{D}$ and can be derived from the same (7). In fact, from equations (7) we can obtain a set of relationships between the sole components D_{ij} of the tensor field $\mathbf{D}$ by elimination of the components $u_{i,j}(\mathbf{x})$ of the gradient $\nabla\mathbf{u}$. These relationships are clearly embedded in the equations (7) and in this sense the solvability of the system can be assumed to be a set of compatibility conditions.

Finally, let us introduce the symmetrical Kirchhoff–Trefftz stress tensor $\mathbf{T}$, related to the Eulerian or Cauchy stress tensor $\mathbf{S}$ by

$$\mathbf{T} = J\mathbf{F}^{-1}\cdot\mathbf{S}\cdot\mathbf{F}^{-T}. \tag{8}$$

The Kirchhoff–Trefftz stress tensor can be associated with a definition of force per unit initial area or volume in the metric induced by the deformation and it is related to the surface traction vector field $\mathbf{f}_0$ and to the body force vector field $\rho_0\mathbf{b}$, both evaluated on the initial configuration, by the equilibrium equations in the Lagrangian or Piola–Signorini form:

$$\operatorname{div}(\mathbf{T}+\nabla\mathbf{u}\cdot\mathbf{T})+\rho_0\mathbf{b} = 0 \quad \text{in } V_0 \tag{9}$$

$$(\mathbf{T}+\nabla\mathbf{u}\cdot\mathbf{T})\cdot\mathbf{n}_0 = \mathbf{f}_0 \quad \text{on } \partial V_0^f \tag{10}$$

ρ_0 is the mass per unit volume in the initial configuration and $\mathbf{b}$ is the vector field of body forces per unit mass. Both $\mathbf{S}$ and $\mathbf{T}$ are supposed to be class C^N ($N \geqslant 2$) tensor fields on the region V_0.

3. COMPLEMENTARY WORK AND EXTENSION OF CROTTI'S THEOREM

It is common knowledge that in the geometrically linearized theory of elasticity Crotti's theorem can be derived straightforwardly from the relationship

$$\mathrm{d}\mathbf{P}^T \cdot \Delta = \int_{V_0} \mathrm{d}\mathbf{S} \times \mathbf{E}\,\mathrm{d}V_0 \tag{11}$$

where $\mathbf{P}$ is the vector of generalised applied forces, $\mathbf{\Delta}$ is the vector of the corresponding generalised displacements and $2\mathbf{E} = (\nabla\mathbf{u}+\nabla\mathbf{u}^T)$ is the infinitesimal strain tensor. As known, in this framework all the vector and tensor fields are understood to be evaluated on the unstrained configuration.

Of course, the existence of a one-to-one correspondence is required between the S_{ij} and the E_{ij} so that, since the expression

$$\mathrm{d}\phi\,(\mathbf{E}) = \mathbf{S} \times \mathrm{d}\mathbf{E} \tag{12}$$

is, for hyperelastic bodies, a total differential, from the expression

$$\mathrm{d}(\mathbf{S} \times \mathbf{E}) = \mathbf{S} \times \mathrm{d}\mathbf{E} + \mathbf{E} \times \mathrm{d}\mathbf{S} \tag{13}$$

we are led to the conclusion that

$$\mathrm{d}\phi_c(\mathbf{S}) = \mathbf{E} \times \mathrm{d}\mathbf{S} \tag{14}$$

is also a total differential (Novozhilov, 1961).

We have therefore the following Euler–Legendre dual transformation (Hill, 1956)

$$\mathbf{S} \times \mathbf{E} - \phi(\mathbf{E}) = \phi_c(\mathbf{S}) \tag{15}$$

and

$$\frac{\partial \phi_c(\mathbf{S})}{\partial S_{ij}} = E_{ij}. \tag{16}$$

This leads immediately to recognition of the right hand member of equation (11) as a perfect differential, so that we can write

$$U_c = \int_{V_0} \phi_c\,(\mathbf{S})\,\mathrm{d}V_0 \tag{17}$$

and consequently we have the classical expression (1) of Crotti's theorem, once provided the relationship $\mathbf{S} = \mathbf{S}(\mathbf{P})$.

Moreover, by commuting differentials into variations δ, equation (11) can be identified with the theorem of virtual forces,

$$\int_{\partial V_0} \delta\mathbf{f}^T \cdot \mathbf{u}\,\mathrm{d}A_0 + \int_{V_0} \rho_0\,\delta\mathbf{b}^T \cdot \mathbf{u}\,\mathrm{d}V_0 = \int_{V_0} \delta\mathbf{S} \times \mathbf{E}\,\mathrm{d}V_0 \tag{18}$$

and consequently Crotti's theorem constitutes a condition of compatibility (Dorn and Schild, 1956).

It is worth observing that the compatibility equations are given by means of the variation of the Eulerian stress field, which is independent of deformation. In fact in the geometrically linearized elasticity theory the Eulerian stress field is always referred to the initial configuration. Finally, there is complete correspondence between the complementary work of applied forces and the complementary work of the internal stress field.

In order to pursue the extension of Crotti's theorem to the finite displacement theory of elasticity it appears natural to assume as starting point the expression of the elementary complementary work of applied forces in the strained configuration. Let us introduce the surface traction vector field $\mathbf{f}$ on ∂V and the body force per unit mass vector field $\mathbf{b}$ on V, both referred to the strained configuration. With reference to the complementary work W_c of applied forces, we can write

$$\mathrm{d}W_c = \int_V \rho\,\mathrm{d}\mathbf{b}^T\cdot\mathbf{u}\,\mathrm{d}V + \int_{\partial V} \mathrm{d}\mathbf{f}^T\cdot\mathbf{u}\,\mathrm{d}A. \tag{19}$$

Let us then consider the Eulerian stress field $\mathrm{d}\mathbf{S}$ in equilibrium with the forces $\rho\,\mathrm{d}\mathbf{b}$ and $\mathrm{d}\mathbf{f}$ according to the Cauchy's equations

$$\widehat{\mathrm{div}}\,\mathrm{d}\mathbf{S} + \rho\,\mathrm{d}\mathbf{b} = 0 \quad \text{in } V \tag{20}$$

$$\mathrm{d}\mathbf{S}\cdot\mathbf{n} = \mathrm{d}\mathbf{f} \quad \text{on } \partial V^f \tag{21}$$

where, of course, the divergence operator $\widehat{\mathrm{div}}$ is defined on $\mathbf{y}$. By means of equations (20) and (21), formulae (A.6) in the Appendix and application of the divergence theorem, we can express the elementary complementary work $\mathrm{d}W_c$ of applied forces in terms of the stress field $\mathrm{d}\mathbf{S}$ as

$$\mathrm{d}W_c = \int_V \rho\,\mathrm{d}\mathbf{b}^T\cdot\mathbf{u}\,\mathrm{d}V + \int_{\partial V} \mathrm{d}\mathbf{f}^T\cdot\mathbf{u}\,\mathrm{d}A = \int_V \mathrm{d}\mathbf{S}\times\hat{\nabla}\mathbf{u}\,\mathrm{d}V. \tag{22}$$

Since in the finite displacement theory of elasticity we cannot assimilate the strained configuration into the initial one, we now perform a change of coordinates from the Eulerian system $\mathbf{y}$ to the Lagrangian one $\mathbf{x}$.

First of all we can observe that equation (10) can be written in the following manner:

$$\mathbf{F}\cdot\mathbf{T}\cdot\mathbf{n}_0 = \mathbf{f}_0 \tag{23}$$

since relationship (2) implies

$$\mathbf{F} = \mathbf{1} + \nabla\mathbf{u}. \tag{24}$$

On account of equations (21), (6), (8) and (23) we have therefore

$$\mathrm{d}\mathbf{f}\,\mathrm{d}A = \mathrm{d}\mathbf{S}\cdot\mathbf{n}\,\mathrm{d}A = J\,\mathrm{d}\mathbf{S}\cdot\mathbf{F}^{-T}\cdot\mathbf{n}_0\,\mathrm{d}A_0 = \mathbf{F}\cdot\mathrm{d}\mathbf{T}\cdot\mathbf{n}_0\,\mathrm{d}A_0 = \mathrm{d}\mathbf{f}_0\,\mathrm{d}A_0. \tag{25}$$

Provided equation (5) and the principle of conservation of mass, that is

$$\rho(\mathbf{y})\,\mathrm{d}V = \rho_0(\mathbf{x})\,\mathrm{d}V_0 \tag{26}$$

the first part of equation (22) becomes

$$\mathrm{d}W_c = \int_{V_0} \rho_0\,\mathrm{d}\mathbf{b}^T\cdot\mathbf{u}\,\mathrm{d}V_0 + \int_{\partial V_0} \mathrm{d}\mathbf{f}_0^T\cdot\mathbf{u}\,\mathrm{d}A_0. \tag{27}$$

Let us now refer the displacement gradients $\hat{\nabla}\mathbf{u}$ of the right hand side of equation (22) to the coordinates $\mathbf{x}$.

We can write

$$\hat{\nabla}\mathbf{u} = \nabla\mathbf{u}\cdot\mathbf{F}^{-1} = \mathbf{F}^{-T}\cdot\mathbf{F}^T\cdot\nabla\mathbf{u}\cdot\mathbf{F}^{-1} \tag{28}$$

and making use of (24) once more,

$$\hat{\nabla}\mathbf{u} = \mathbf{F}^{-T}\cdot(\mathbf{1}+\nabla\mathbf{u})^T\cdot\nabla\mathbf{u}\cdot\mathbf{F}^{-1} = \mathbf{F}^{-T}\cdot(\nabla\mathbf{u}+\nabla\mathbf{u}^T\cdot\nabla\mathbf{u})\cdot\mathbf{F}^{-1}. \tag{29}$$

On account of equation (29), let us now write the internal product of the right hand side of equation (22)

$$\mathrm{d}\mathbf{S}\times\hat{\nabla}\mathbf{u} = \mathrm{d}\mathbf{S}\times[\mathbf{F}^{-T}\cdot(\nabla\mathbf{u}+\nabla\mathbf{u}^T\cdot\nabla\mathbf{u})\cdot\mathbf{F}^{-1}] = \mathrm{tr}\,[\mathrm{d}\mathbf{S}\cdot\mathbf{F}^{-T}\cdot(\nabla\mathbf{u}+\nabla\mathbf{u}^T\cdot\nabla\mathbf{u})\cdot\mathbf{F}^{-1}]. \tag{30}$$

Making reference to formulae (A.3) in the Appendix and by definition (8) we get

$$\mathrm{d}\mathbf{S}\times\hat{\nabla}\mathbf{u} = \mathrm{tr}\,[\mathbf{F}^{-1}\cdot\mathrm{d}\mathbf{S}\cdot\mathbf{F}^{-T}\cdot(\nabla\mathbf{u}^T+\nabla\mathbf{u}^T\cdot\nabla\mathbf{u})] = \mathrm{tr}\,[\mathrm{d}\mathbf{T}\cdot(\nabla\mathbf{u}^T+\nabla\mathbf{u}^T\cdot\nabla\mathbf{u})]. \tag{31}$$

Finally, the symmetry of the Kirchhoff–Trefftz stress tensor $\mathrm{d}\mathbf{T}$ leads to

$$\mathrm{d}\mathbf{S}\times\hat{\nabla}\mathbf{u} = \mathrm{tr}\,[\tfrac{1}{2}\mathrm{d}\mathbf{T}\cdot(\nabla\mathbf{u}+\nabla\mathbf{u}^T+2\nabla\mathbf{u}^T\cdot\nabla\mathbf{u})] = \mathrm{d}\mathbf{T}\times(\mathbf{D}+\tfrac{1}{2}\nabla\mathbf{u}^T\cdot\nabla\mathbf{u}). \tag{32}$$

So far the proposed change of coordinates in equation (22) is completed:

$$\mathrm{d}W_c = \int_{V_0}\rho_0\,\mathrm{d}\mathbf{b}^T\cdot\mathbf{u}\,\mathrm{d}V_0+\int_{\partial V_0}\mathrm{d}\mathbf{f}_0^T\cdot\mathbf{u}\,\mathrm{d}A_0 = \int_{V_0}\mathrm{d}\mathbf{T}\times(\mathbf{D}+\tfrac{1}{2}\nabla\mathbf{u}^T\cdot\nabla\mathbf{u})\,\mathrm{d}V_0. \tag{33}$$

In this expression we can immediately note that the non-linearity of the problem is the origin of the term $\frac{1}{2}\,\mathrm{d}\mathbf{T}\times(\nabla\mathbf{u}^T\cdot\nabla\mathbf{u})$, and therefore there is not a direct correspondence between the complementary work of applied forces and the complementary work of the stress field, contrary to what happens in the geometrically linearized theory. This means that the equivalence in energy of the work performed by applied and internal forces, which relies on the principle of conservation of energy, is not followed by the equivalence of the respective Euler–Legendre transformations. Nevertheless, let us point out the circumstances under which the form in which we have arranged the internal stress member of equation (33) can represent a perfect differential.

First of all it is clear that in this framework also we can suppose that the components of the Lagrangian stress field $\mathbf{T}$ are single-valued functions of the components of the Green strain tensor $\mathbf{D}$ satisfying

$$\frac{\partial T_{ij}}{\partial D_{mn}} = \frac{\partial T_{mn}}{\partial D_{ij}} \tag{34}$$

and we have everywhere in the hyperelastic body

$$\frac{\partial\Phi(\mathbf{D})}{\partial D_{ij}} = T_{ij}. \tag{35}$$

Furthermore, if we suppose that the constitutive equations (35) establish a one-to-one correspondence between the T_{ij} and the D_{ij} in this case we can also admit the existence of the following Euler–Legendre dual transformation (Ogden, 1984)

$$\mathbf{T}\times\mathbf{D}-\Phi(\mathbf{D}) = \Phi_c(\mathbf{T}) \tag{36}$$

and the definition of complementary energy density $\Phi_c(\mathbf{T})$ yields

$$\frac{\partial\Phi_c(\mathbf{D})}{\partial T_{ij}} = D_{ij}. \tag{37}$$

However, given the invertibility of relationships (35) and the solvability of the equations (7), as discussed in Section 2, we note that the last term of equation (33) represents a perfect differential only if

$$dG = \tfrac{1}{2}d\mathbf{T} \times (\nabla\mathbf{u}^T \cdot \nabla\mathbf{u}) \tag{38}$$

is also a perfect differential.

From a physical point of view, it is worth observing that this condition is due to the fact that equation (33) keeps 'memory' of the actual transformation and this fact cannot be yielded on the whole by the energy statement (37), contrary to what happens in the geometrically linearized theory of elasticity. Moreover, in general it is a difficult matter to determine the conditions for the integrability of expression (38) and it appears preferable to investigate the distinctive features of the problem at hand. Nevertheless, when the case is that (38) is a perfect differential, we can write

$$U_c^* = \int_{V_0} \Phi_c(\mathbf{T}) + G(\mathbf{T})\, dV_0 \tag{39}$$

and this yields the natural extension of Crotti's theorem to the finite displacements theory of elasticity, provided we express the functional relationship (33) in terms of generalised forces and displacements such that

$$d\mathbf{P}^T \cdot \mathbf{\Delta} = dU_c^*(\mathbf{P}) \tag{40}$$

and therefore

$$\frac{\partial U_c^*}{\partial P_p} = \Delta_p. \tag{41}$$

It is straightforward to verify that equation (33) and, consequently, equations (40) and (41) naturally reduce themselves to the expressions (18), (11) and (1) when we consider the displacement gradients as negligible before unity ($|\nabla\mathbf{u}| \ll 1$) and the stress field is evaluated under the conditions $\mathbf{x} \approx \mathbf{y}$, as is usual in the geometrical linearized theory of elasticity.

4. EXTENDED CROTTI'S THEOREM AS A COMPATIBILITY CONDITION

To complete our discourse it remains to show how the conditions of compatibility and the boundary kinematic conditions in the finite displacement theory of elasticity are embedded in equation (33).

In order to do so we partly follow a classical variational procedure (Friedrichs, 1929), which has been often employed to various degrees in the literature dealing with variational principles for finite elastic displacements (Reissner, 1953; Baldacci, 1967; Fraeijs de Veubeke, 1972; Washizu, 1975; Oden and Reddy, 1983).

Retracing the steps which led from (19) to (33), it is straightforward to verify that in this case we can also formally commute differentials $d\mathbf{b}$, $d\mathbf{f}$ and $d\mathbf{T}$ into variations $\delta\mathbf{b}$, $\delta\mathbf{f}$ and $\delta\mathbf{T}$. This means that, according to the same procedure, the following relationship:

$$\int_{V_0} \rho_0\, \delta\mathbf{b}^T \cdot \mathbf{u}\, dV + \int_{\partial V_0} \delta\mathbf{f}_0^T \cdot \mathbf{u}\, dA_0 = \int_{V_0} \delta\mathbf{T} \times (\mathbf{D} + \tfrac{1}{2}\nabla\mathbf{u}^T \cdot \nabla\mathbf{u})\, dV_0 \tag{42}$$

holds true, provided the variation of the Lagrangian stress field $\delta\mathbf{T}$ is related to the variation of applied forces $\rho\delta\mathbf{b}$ and $\delta\mathbf{f}$ by means of the equations

$$\operatorname{div}(\delta\mathbf{T} + \nabla\mathbf{u} \cdot \delta\mathbf{T}) + \rho_0 \delta\mathbf{b} = 0 \quad \text{in } V_0 \tag{43}$$

$$(\delta\mathbf{T} + \nabla\mathbf{u} \cdot \delta\mathbf{T}) \cdot \mathbf{n}_0 = \delta\mathbf{f}_0 \quad \text{on } \partial V_0^f. \tag{44}$$

As a matter of fact equation (42) results to be equivalent to the stationary value of the functional

$$\Psi = \int_{V_0} \rho_0 \mathbf{b}^T \cdot \mathbf{u} \, \mathrm{d}V + \int_{\partial V_0} \mathbf{f}_0^T \cdot \mathbf{u} \, \mathrm{d}A_0 - \int_{V_0} [\Phi_c(\mathbf{T}) + \tfrac{1}{2}\mathbf{T} \times (\nabla\mathbf{u}^T \cdot \nabla\mathbf{u})] \, \mathrm{d}V_0 \tag{45}$$

with respect to all the equilibrated variations of the static fields.

For the sake of generality we can also consider the following boundary kinematic conditions imposed on the displacement field:

$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{on } \partial V_0^u \tag{46}$$

provided the boundary surface ∂V_0 is partitioned as follows: $\partial V_0 = \partial V_0^u \cup \partial V_0^f$. Forces per unit surface are applied on ∂V_0^f.

Now we render explicit the compatibility conditions embedded in the variational statement (42) by subtracting from the functional (45) the Piola–Signorini equilibrium equations (9) and (10) through the Lagrangian multipliers fields $\mathbf{h}$ and $\mathbf{k}$:

$$\Psi^* = \int_{V_0} \rho_0 \mathbf{b}^T \cdot \mathbf{u} \, \mathrm{d}V + \int_{\partial V_0} \mathbf{f}_0^T \cdot \mathbf{u} \, \mathrm{d}A_0 - \int_{V_0} [\Phi_c(\mathbf{T}) + \tfrac{1}{2}\mathbf{T} \times (\nabla\mathbf{u}^T \cdot \nabla\mathbf{u})] \, \mathrm{d}V_0$$
$$- \int_{V_0} [\operatorname{div}(\mathbf{T} + \nabla\mathbf{u} \cdot \mathbf{T}) + \rho_0 \mathbf{b}]^T \cdot \mathbf{h} \, \mathrm{d}V_0 - \int_{\partial V_0^f} [(\mathbf{T} + \nabla\mathbf{u} \cdot \mathbf{T}) \cdot \mathbf{n}_0 - \mathbf{f}_0]^T \cdot \mathbf{k} \, \mathrm{d}A_0. \tag{47}$$

The functional Ψ^* is now stationary with regard to variations of the static fields as well as to variations of the Lagrangian multiplier field. It is worth noting that the $\delta\mathbf{f}$, the $\delta\mathbf{b}$ and the $\delta\mathbf{T}$ no longer need to satisfy the set of equations (43) and (44) *a priori*.

At this point the natural conditions or Euler equations of the problem are yielded by free variations of the static fields (Courant and Hilbert, 1953).

In fact we can derive from (47)

$$\delta\Psi^* = \int_{\partial V_0^u} \delta\mathbf{f}_0^T \cdot \bar{\mathbf{u}} \, \mathrm{d}A_0 + \int_{\partial V_0^f} \delta\mathbf{f}_0^T \cdot \mathbf{k} \, \mathrm{d}A_0 - \int_{\partial V_0^f} \{[(\delta\mathbf{T} + \nabla\mathbf{u} \cdot \delta\mathbf{T}) \cdot \mathbf{n}_0]^T \cdot \mathbf{k} - \delta\mathbf{f}_0^T \cdot \mathbf{u}\} \, \mathrm{d}A_0$$
$$+ \int_{V_0} \rho_0 \delta\mathbf{b}^T \cdot (\mathbf{u} - \mathbf{h}) \, \mathrm{d}V_0 - \int_{V_0} \{\delta\mathbf{T} \times \mathbf{D} + \tfrac{1}{2}\delta\mathbf{T} \times (\nabla\mathbf{u}^T \cdot \nabla\mathbf{u})$$
$$+ [\operatorname{div}(\delta\mathbf{T} + \nabla\mathbf{u} \cdot \delta\mathbf{T})]^T \cdot \mathbf{h}\} \, \mathrm{d}V_0 = 0 \tag{48}$$

and by means of formulae (A.6) in the Appendix and the divergence theorem we get

$$\delta\Psi^* = \int_{\partial V_0^u} \delta\mathbf{f}_0^T \cdot (\bar{\mathbf{u}} - \mathbf{h}) \, \mathrm{d}A_0 + \int_{\partial V_0^f} \delta\mathbf{f}_0^T \cdot (\mathbf{k} - \mathbf{h}) \, \mathrm{d}A_0$$
$$+ \int_{\partial V_0^f} \delta\mathbf{f}_0^T \cdot (\mathbf{u} - \mathbf{k}) \, \mathrm{d}A_0 + \int_{V_0} \rho_0 \delta\mathbf{b}^T \cdot (\mathbf{u} - \mathbf{h}) \, \mathrm{d}V_0$$
$$- \int_{V_0} \delta\mathbf{T} \times (\mathbf{D} - \nabla\mathbf{h} - \nabla\mathbf{h}^T \cdot \nabla\mathbf{u} + \tfrac{1}{2}\nabla\mathbf{u}^T \cdot \nabla\mathbf{u}) \, \mathrm{d}V_0 = 0. \tag{49}$$

As variations $\delta\mathbf{f}_0$, $\delta\mathbf{b}_0$ and $\delta\mathbf{T}$ are totally arbitrary, $\delta\Psi^*$ vanishes if and only if

$$\begin{aligned} \bar{\mathbf{u}} &= \mathbf{h} \quad \text{on } \partial V_0^u \\ \mathbf{k} &= \mathbf{h} \quad \text{on } \partial V_0^f \\ \mathbf{u} &= \mathbf{k} \quad \text{on } \partial V_0^f \\ \mathbf{k} &= \mathbf{h} \quad \text{in } V_0 \\ \mathbf{D} &= \tfrac{1}{2}(\nabla\mathbf{h} + \nabla\mathbf{h}^T + 2\nabla\mathbf{h}^T \cdot \nabla\mathbf{u} - \nabla\mathbf{u}^T \cdot \nabla\mathbf{u}) \quad \text{in } V_0. \end{aligned} \tag{50}$$

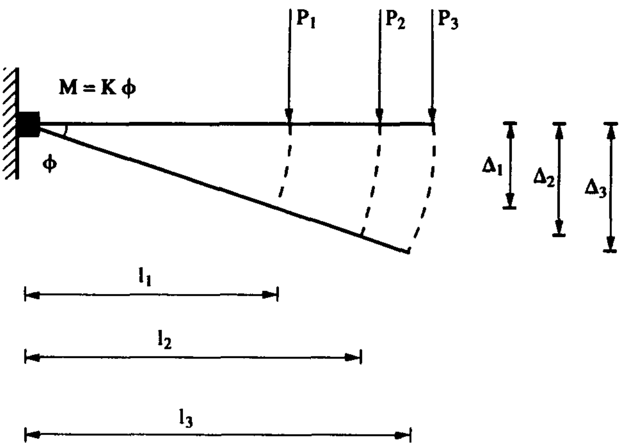

Fig. 1.

Therefore the variational expression obtained gives the physical meaning of the Lagrangian multiplier fields **h** and **k**, which can be identified with the displacement field **u**, and returns the compatibility conditions (7) and the boundary kinematic conditions (46) as the Euler equations of the problem.

It is thus clear that the relationship (42) gives origin to the set of conditions of compatibility in the finite displacement theory of elasticity in the sense discussed in Section 2 and these conditions are as well embedded in the extended formulation of Crotti's theorem.

5. A SIMPLE APPLICATION

So far we have established the possibility of extending Crotti's theorem to the finite displacement theory of elasticity. When this is possible, the main operational difficulty remains, of course, in integrating expression (39).

We now consider what is perhaps the simplest case in the theory of structures in order to illustrate the process in terms of generalised forces and displacements.

Let us examine the elastically hinged beam of Fig. 1. The Lagrangian parameter is naturally the angle of rotation ϕ and we can restrict our attention to the set $\phi \in [0, \pi/2[$ without loss of generality. The generalised external actions are the applied vertical forces P_i, the corresponding generalised displacements are the vertical displacements Δ_i and the generalised internal force is the moment M acting on the rotational spring.

For the sake of simplicity, we consider the constitutive relationship be given by

$$M = k\phi \tag{51}$$

where k is a constant.

In the geometrically linearized theory of elasticity the complementary energy of internal forces, whose derivative yields the classical Crotti's theorem, is given by

$$\mathrm{d}U_c(M) = \mathrm{d}(M\phi) - M\,\mathrm{d}\phi = \phi\,\mathrm{d}M = \frac{M}{k}\mathrm{d}M \tag{52}$$

and therefore, apart from a constant,

$$U_c(M) = \frac{1}{2}\frac{M^2}{k}. \tag{53}$$

In fact, as in this framework we have as equilibrium equation

$$M = P_i l_i \tag{54}$$

we get the following expression for the classical Crotti's theorem

$$\frac{\partial U_c}{\partial P_i} = \frac{\partial U_c}{\partial M}\frac{\partial M}{\partial P_i} = \frac{M}{k} l_i = \phi l_i = \Delta_i \tag{55}$$

which represents, as we consider $\sin\phi \approx \phi$, a compatibility condition.

Let us now move to the finite displacements theory of elasticity. In accordance with the theory developed in the previous paragraphs, we have to express the complementary work of external forces in terms of the internal force M.

Given the equilibrium equation

$$M = P_i l_i \cos\phi \tag{56}$$

we have, without any geometrical restriction,

$$\mathrm{d}P_i \Delta_i = \mathrm{d}P_i l_i \sin\phi = \frac{\mathrm{d}M}{l_i \cos\phi} l_i \sin\phi = \mathrm{d}M \tan\phi. \tag{57}$$

By inversion of relationship (51) we can integrate the term $\mathrm{d}U_c^*$ which is reduced to a function of generalised internal force M.

Therefore, we get, apart from a constant,

$$U_c^*(M) = k \cdot \ln\left(\sec\left(\frac{M}{k}\right)\right). \tag{58}$$

With reference to expressions (58) and (53), it is evident that in the limit we have, according to l'Hôpital's rule,

$$\lim_{\phi\to 0} M(\phi) = 0 \quad \Rightarrow \quad \lim_{\phi\to 0}\frac{U_c^*}{U_c} = \frac{k \cdot \ln\left(\sec\left(\frac{M}{k}\right)\right)}{\frac{1}{2}\frac{M^2}{k}} = 1 \tag{59}$$

as pointed out at the end of Section 3.

However, in the finite displacements theory of elasticity the derivatives of expression (58) can provide the generalised displacements corresponding to the generalised forces P_i in full generality:

$$\frac{\partial U_c^*}{\partial P_i} = \frac{\partial U_c^*}{\partial M}\frac{\partial M}{\partial P_i} = \frac{\mathrm{d}\left(k \cdot \ln\left(\sec\left(\frac{M}{k}\right)\right)\right)}{\frac{\mathrm{d}M}{l_i \cos\phi}} = l_i \sin\phi = \Delta_i \tag{60}$$

and therefore constitute the extension of expression (55) to the finite displacements theory of elasticity in the spirit of classical Crotti's theorem.

6. CONCLUSIONS

In the present paper the extension of Crotti's theorem to the finite displacement theory of elasticity has been examined.

Firstly, it has been shown how in a Lagrangian framework this result can be obtained by the direct analysis of the expression corresponding to the complementary work of applied forces in terms of internal stresses and strains, provided certain additional conditions are satisfied with respect to the classical formulation of the theorem in the geometrically linearized theory of elasticity. Secondly, it has been proved that the conditions of compatibility and the boundary kinematic conditions in the theory of finite elastic displacements are implicitly embedded in the extended formulation of the theorem.

The result appears interesting from a theoretical point of view and clarifies some questions which seem still unresolved in available literature. However, it is worth noting that the non-linearity of the problem considerably complicates the matter and with regard to practical cases it will be difficult to perform the required integrations in a closed form. Nevertheless the procedure can suggest the possibility of deriving or imposing specific conditions of compatibility in suitable cases and can provide a complementary approach to some techniques which are based on strain energy.

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APPENDIX

We define the product $\mathbf{A} \cdot \mathbf{B}$ of two first or second-order tensors $\mathbf{A}$ and $\mathbf{B}$ in the following manner:

$$(\mathbf{A} \cdot \mathbf{B})_{ij} = A_{ik} B_{kj}. \tag{A.1}$$

Given a second-order tensor $\mathbf{A}$ we call $\mathbf{A}^T$ the transpose of $\mathbf{A}$ and we have

$$(\mathbf{A}^T)_{ij} = A_{ji}. \tag{A.2}$$

The inner product $\mathbf{A} \times \mathbf{B}$ of two second order tensors $\mathbf{A}$ and $\mathbf{B}$ is defined by

$$\mathbf{A} \times \mathbf{B} = \mathrm{tr}\,[\mathbf{A}^T \cdot \mathbf{B}] = \mathrm{tr}\,[\mathbf{B}^T \cdot \mathbf{A}] = \mathrm{tr}\,[\mathbf{A} \cdot \mathbf{B}^T] = \mathrm{tr}\,[\mathbf{B} \cdot \mathbf{A}^T] = A_{ij} B_{ij}. \tag{A.3}$$

If $\mathbf{u}$ is a vector field on a region Ω and is differentiable on Ω (i.e. at each point $\mathbf{x} \in \Omega$), we call gradient of $\mathbf{u}$ at $\mathbf{x}$ the following expression:

$$\nabla \mathbf{u} = \frac{\partial u_i(\mathbf{x})}{\partial x_j} = u_{i,j}(\mathbf{x}). \tag{A.4}$$

The divergence of a differentiable tensor field $\mathbf{A}$ on a region Ω is defined by

$$[\mathrm{div}\,\mathbf{A}(\mathbf{x})]_i = A_{ij,j}(\mathbf{x}). \tag{A.5}$$

If $\mathbf{u}$ is a vector field and $\mathbf{A}$ is a tensor field, both of class C^2 on Ω, the following relationship holds true:

$$\mathrm{div}\,(\mathbf{A}^T \cdot \mathbf{u}) = \mathrm{div}\,\mathbf{A}^T \cdot \mathbf{u} + \mathbf{A} \times \nabla \mathbf{u}. \tag{A.6}$$